

The Log Contract

A new instrument to hedge volatility.

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The derivatives markets have grown enormously in recent years. According to the Bank for International Settlements [1992], the notional principal value of the over-the-counter option positions outstanding at the end of 1991 amounted to \$630 billion. This represents a formidable amount of risk, which has to be managed and controlled.

Ever since Black and Scholes [1973] wrote their article on option pricing twenty years ago, the basic principles of delta-hedging have been well-understood and widely practiced. An option writer can hedge against changes in the value of the underlying asset by taking a suitable position in the underlying market.

But delta-hedging does not eliminate all risk. The major risk that remains is volatility risk. A trader who has written a large number of options will lose money if volatility suddenly rises, even if the portfolio is properly delta-hedged.

Volatility has to be managed. Investors and traders must manage their portfolios to reflect their views on future volatility. If they have no views on volatility, they must try to control and minimize their volatility exposure.

Our knowledge of how to hedge volatility has been improving. While the original Black-Scholes model of option pricing assumes that volatility is constant, in recent years a number of researchers have developed option pricing models that recognize the stochastic nature of volatility (see, for example, Hull

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and White [1987]). New econometric techniques have made it possible to explore the empirical behavior of volatility in a more sophisticated manner.

But the instruments for implementing a hedging strategy have remained rather crude. This article sets out the case for a new instrument — the Log Contract — which would provide an accurate and flexible volatility hedge.

The Log Contract is a futures style contract whose settlement price is equal to the logarithm of the price of the underlying asset. As is shown below, the Log Contract can be used to provide a payoff that depends only on the difference between the volatility expected at the time the contract was entered into and the actual volatility that occurs over its life.

WHY HEDGE VOLATILITY?

The reason why volatility hedging is so important is that, once delta risk is removed, volatility is the major source of residual risk. To see this, think of a bank that writes a one-month European call giving its customer the option to buy £1m at today's one-month dollar/sterling forward rate. If the bank does not hedge at all, it faces substantial currency exposure. Sterling could well appreciate by 5% over the month, leaving the bank with a loss equal to two or three times the option premium. On the other hand, if sterling depreciates, the option will expire unexercised, and the bank will make a profit equal to the premium.

Writing naked options is obviously a risky business. If the market is efficient, the bank can expect to break even more or less in the long run, but each time it writes an option it is exposed to a risk of the same order as the option premium itself.

In practice, the bank will seek to reduce its risk by hedging. As sterling appreciates, the probability of the option ending up in the money increases, and the bank's expected liability grows. To hedge against this, the bank can take a long position in sterling. The precise size of the hedge can be calculated using the Black-Scholes formula. If the assumptions underlying the formula hold, the hedge will work perfectly and the bank will bear no risk.

How well does the Black-Scholes hedge work in practice? Exhibit 1 shows the result of a paper exercise using actual exchange rate data for five years beginning June 1, 1987. The average volatility of the dollar/sterling exchange rate over the period was

EXHIBIT 1

Profit on Writing \$/£ Call Options as a Proportion of the Option Premium — Distribution of Outcomes Monthly 1987-1992

	Naked	Delta-Hedged
Mean	-29%	2%
Standard Deviation	171%	32%
Interquartile Range	from -200% to +100%	from -23% to +24%

10.8%. Every month for sixty months, the bank writes a one-month at-the-money call option on £1m at an implied volatility of 10.8%. At the end of each month it measures its profit or loss and divides it by the option premium.

The first column of Exhibit 1 shows how risky it is to write naked options. Over the five-year period, the bank would have made a net loss equal to 29% of the total premium received. It would have been unlucky. In the long run, it could expect to come close to break-even. But sixty months is too short a period, given the substantial variability in outcome from month to month.

The riskiness is illustrated in the second row; the standard deviation of the profit from the naked call strategy is equal to 171% of the option premium. 25% of the time the bank made a loss of 200% or more of its premium. One-quarter of the time the option expired worthless, and the bank captured the entire premium.

The second column shows what happens if the bank delta-hedges. The hedge is rebalanced daily using the Black-Scholes formula. The average monthly profit is now close to zero. The standard deviation is only 32% of the option premium. Delta-hedging has reduced the bank's risk by a factor of five.

The theory says that the bank's risk should be eliminated completely by delta-hedging. Why does some risk remain? Option pricing theory makes a number of assumptions that are violated in practice — that the portfolio is rebalanced continuously, that the changes in the exchange rate are distributed lognormally, that the rate does not jump, and that volatility is known and constant. Each of these is violated in practice, and each contributes to the residual risk of a delta-hedged portfolio. But it is the fact that the outcome volatility over a month differs from the bank's forecast that accounts for the great bulk of the hedge error.

**EXHIBIT 2
DOLLAR STERLING EXCHANGE RATE
VOLATILITY OF DAILY RETURNS**

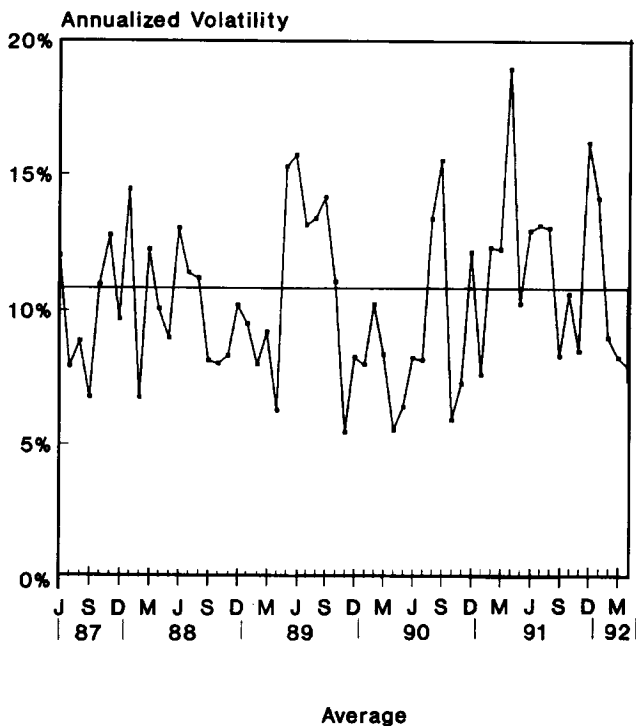


Exhibit 2 graphs the actual volatility each month over the five-year period.² It was quite variable. It averaged 10.8% but went as low as 5.4% and as high as 18.9%. Exhibit 3 shows that most of the hedge error each month can be explained by the variations in the outcome volatility. In those months when the actual volatility was much higher than the 10.8% the bank assumed when writing and hedging the option, the bank generally lost money. Conversely, when volatility was below the forecast level, the bank made money.

The regression of the percentage hedge error (%HE) on the outcome volatility (σ) gives the following equation:

$$\%HE = -9.64\sigma + 1.016 \quad (R^2 = 0.815)$$

The correlation coefficient R^2 shows that over 80% of the hedging error that remains after delta-hedging is due to an incorrect forecast of the volatility over the life of the option. Delta-hedging reduces hedge errors by a factor of five; volatility hedging could potentially reduce hedge errors by a further factor of five.

If the bank cannot forecast volatility better than the market, it should attempt to minimize its exposure to volatility. In this way it will be able to minimize its risk and be able to write a much greater volume of business for a given commitment of risk capital. But to do this effectively it needs to be able to hedge volatility in a cheap and accurate manner.

Options writers are not the only people who need good volatility hedges. Portfolio insurers would also find them valuable. If they seek to insure some floor value to their portfolio by trading index futures, they are effectively creating a synthetic option position. To the extent that volatility turns out to be higher than they forecast at the outset, they will find themselves paying an unexpectedly high price for the insurance. An instrument that hedges volatility would enable the investor to lock in the cost of the insurance.

A straightforward mechanism for hedging volatility would also be useful for traders who want to take an outright position on volatility. If an option is trading on an implied volatility of 10%, but the trader expects the volatility to go up to 15%, the option looks cheap and the trader will buy. The trader is using the options market to take a position on volatility.

But Exhibit 3 shows that the relation between the payoff to a delta-hedging strategy and volatility, though good, is not perfect. The trader could be right about future volatility, carry out the strategy, and still lose money. It would be better to trade an instrument whose value is directly related to volatility.

INSTRUMENTS FOR HEDGING VOLATILITY

If there is such a widespread need for a good volatility hedge, why does no suitable instrument exist? One reason could be that traders can already use a delta-hedged option position as a position on volatility — as we have seen, the payoff to a delta-hedged option is quite well correlated with volatility. Options writers can immunize their books to volatility shifts by maintaining a “gamma-neutral” position; the portfolio insurer can buy and delta-hedge short-dated options to hedge against volatility shocks; a speculator can take a position on volatility by setting up an option position and delta-hedging it.

But this is not wholly convincing as an explanation for the lack of a direct contract on volatility. A

delta-hedged option is not a perfect volatility play. For hedging and risk control purposes, a correlation of 0.815 between the quantity being hedged (volatility) and the hedging instrument (a delta-hedged option) looks very low. The existence of an inadequate substitute should not prevent the development of a better instrument. Most instruments used for risk management — for example, swaps and futures — have developed precisely because they provide a better, cheaper, or more direct way of doing something that could already be done less well using the cash market.

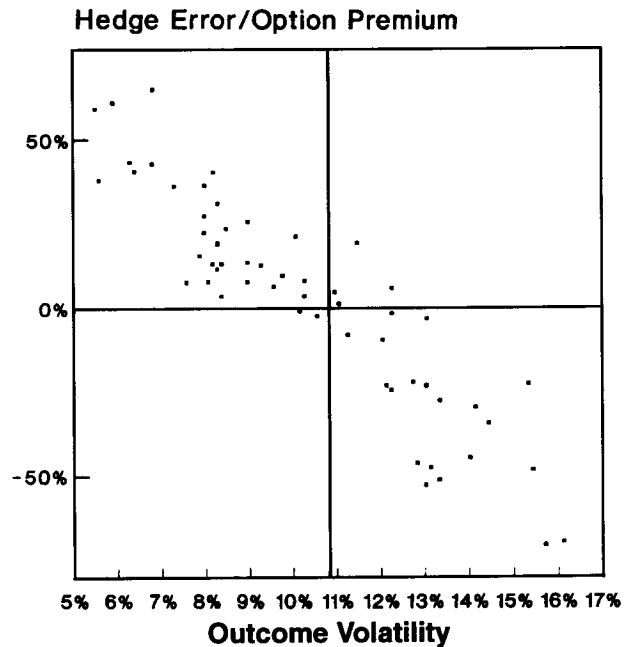
A contract on the implied volatilities of traded options, along the lines being proposed by the Chicago Board Options Exchange, might go some way toward meeting the need for a volatility hedge. But changes in actual volatility and changes in the implied volatility in option prices are quite distinct. An implied volatility contract does not provide a good hedge against actual volatility.

A possible explanation for the lack of a direct volatility contract is that it is difficult to devise a robust instrument that is not subject to manipulation. Suppose, for example, we want to set up a volatility contract on a stock index. One would start from a liquid futures contract such as the S&P 500. The corresponding volatility contract could be a futures style contract that expires on the same day as the underlying future. The settlement price of the volatility contract at expiration would be equal to the actual volatility of the underlying futures price over its life, based on daily settlement prices. Prior to maturity, the price of the volatility contract would be determined by market supply and demand just as with any other futures contract.

Such a contract might be particularly subject to manipulation, which is an issue for any derivative contract. By trading heavily on the spot market, it is possible to shift the spot price and hence manipulate the payoff on the derivative contract. In ordinary futures contracts the damage is self-limiting.

Suppose, for example, I am long \$1m in the futures market, and each purchase of \$1m in the spot market moves the spot price (and hence the futures price) 1 tick. If I buy \$20m in the spot market, I will move the price 20 ticks, and be able to liquidate my futures contract at a price 20 ticks above the true price. If the market is illiquid, however, the sale of the futures contract will depress the price, so when I

**EXHIBIT 3
HEDGE ERROR IS CORRELATED WITH
OUTCOME VOLATILITY**



Positions rebalanced daily
Forecast volatility of 10.8%

come to sell back my spot purchases I will lose on the spot market what I have gained on the futures market.

In the case of the volatility contract, manipulation is much cheaper. Suppose I am long a one-month volatility contract. On a day when the futures price has moved up a lot, I buy in the futures market to move the futures price still farther, and sell it back the following day. Conversely, when the market has moved down, I sell and buy back the following day. The cost of doing this is low, since I am not trying to shift a market a large amount, but merely to add volatility to the price path. So manipulation may be a serious problem for a volatility contract.

A direct volatility futures contract also suffers from inflexibility. I have suggested that it be calculated using daily closing prices, but some investors may be interested in the volatility of opening prices or mid-day prices, or they may be more interested in the volatility of hourly or weekly returns. A single volatility futures contract could hedge only one definition of volatility.

THE LOG CONTRACT

There is a contract that would be no easier to manipulate than a conventional futures contract, and that would enable traders to hedge volatility accurately using their own preferred periods for calculating returns. The contract is a futures-style contract that is tied to a conventional futures contract. The two expire at the same time. If the conventional futures settlement price at expiration is F_T , the settlement price of the Log Contract, L_T , is defined to be equal to $\text{Log}(F_T)$, where Log is simply the natural logarithm.

Suppose, for example, the aim is to provide a hedge for volatility of the dollar/sterling exchange rate over the next month, and that there is a futures contract that expires in a month. If the final settlement price of the futures contract is \$2.00/£1, the final settlement price of the Log Contract would be $\text{Log } 2.00$ or 0.693.

Using the standard Black-Scholes assumptions, in particular that the forward price follows a geometric Brownian process with constant volatility σ , it can be shown that the fair price of the Log Contract at time t , L_t , is given by:

$$L_t = \text{Log}F_t - \frac{1}{2} \sigma^2 (T - t)$$

where F_t is the futures price at time t .

With a conventional put or call option, the hedge ratio or delta depends on the level of future volatility assumed. If the hedger uses an incorrect estimate of volatility, the investment will not be properly hedged. The Log Contract is different. The delta is equal to $+1/F_t$; a trader who is long 1 Log Contract will delta-hedge by shorting \$1 worth of the underlying asset.

The delta does not depend on the forecast level of volatility. The hedger does not need to forecast volatility correctly in order to delta-hedge accurately. This is what ensures that the performance of a delta-hedged Log Contract depends only on the outcome volatility and not on the hedger's forecast of volatility.

Suppose that the trader shorts 1 Log Contract and delta-hedges by going long sterling to the value of \$1. If sterling appreciates, the price of the Log Contract will tend to rise, and so will the sterling futures price. The trader's position in the sterling

futures market will now be worth more than \$1, so to keep the hedge intact the trader has to sell sterling futures. Conversely, as sterling depreciates, the trader buys sterling futures.

In the appendix it is proved that the present value of the trader's profit from following this strategy over the life of the volatility contract is almost exactly equal to:

$$\frac{1}{2} (\text{ISD}^2 - \sigma^2) T$$

where σ is the outcome volatility, and ISD is the volatility implied in the price of the Log Contract.

A Log Contract can be delta-hedged without making any forecast of volatility, and the hedged position is a pure and simple volatility play. This result is not dependent on the assumption that returns are generated by a Brownian diffusion process, or that volatility is constant. The proof in the appendix assumes only that prices do not move too much each time the hedge is rebalanced.

A test of how well the result holds in practice is set out in Exhibit 4. This is the same as Exhibit 3, and continues to show the profit or loss from writing delta-hedged calls (as shown by the dots). In addition, the Exhibit superimposes the performance of a portfolio consisting of a delta-hedged Log Contract for each of the sixty months. The results are shown as crosses. The crosses lie almost exactly on the curve with virtually no deviation on either side. They show that the profit from a strategy of buying a Log Contract and delta-hedging it depends purely on the outcome volatility.

Regressing the hedge error against the squared outcome volatility gives:

$$\%HE = -43.1\sigma^2 + 0.500 \quad (R^2 = 0.99987) \quad (4)$$

With a correlation coefficient of 99.99%, the fit is virtually exact. The delta-hedged Log Contract provides a near-perfect volatility hedge.

The calculation uses actual daily closing prices. It does not assume continuous rebalancing; positions are adjusted just once each trading day of the month. It does not assume that there were no jumps. On individual days the change in the exchange rate was as large as 2.8%. It does not assume that returns are distributed lognormally; in fact the coefficient of

kurtosis is 1.60, suggesting (as many other studies have shown) that the distribution of exchange rate returns is fat-tailed.

The hedge error is calculated assuming that the portfolio is rebalanced at daily closing prices. Volatility is calculated using the same prices. Outcome volatility is sensitive to the timing and period used for calculating returns.

For example, we could have calculated monthly volatility of the sterling/dollar exchange rate using two-day returns rather than one-day returns. The two estimates will be similar but not identical. In our data set, the root mean square difference between the two was 1.13%, so if the volatility in a month estimated using daily returns was 10%, the estimate using two-day returns might easily be 11.13% or 8.87%. This difference is clearly significant from a trader's perspective.

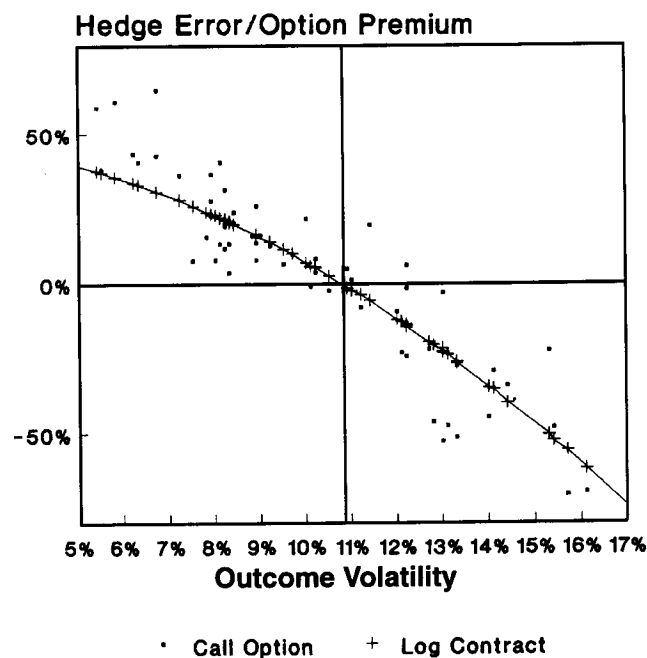
One attraction of using the Log Contract to hedge volatility is that traders can decide for themselves the period over which they wish to calculate returns. For example, a trader may be concerned that using daily returns to calculate volatility produces significant sample error — in a month there are only twenty-two observations. If a position on the volatility of hourly returns is preferable, the trader can do this by rebalancing the hedge on the Log Contract every hour. Similarly, closing prices are not necessary as the basis for the definition of volatility; by rebalancing at noon for example, the trader can create an exposure to the volatility of noon-to-noon returns.

CONCLUSIONS

There is a need for an instrument to hedge volatility. A straight volatility futures contract is too easy to manipulate. But a Log Contract is a good alternative. It cannot be manipulated easily. When delta-hedged, it provides a pure play on volatility. The same instrument can be used for hedging volatility whatever the period used by the trader for defining returns.

The instrument has another attractive feature. The most liquid call and put options are those that are close to the money. As time evolves, an option that was close to the money and liquid will tend to go deep in or out of the money. It will tend to become illiquid. It will also tend to lose its option characteristics and to trade solely on its intrinsic value. If it goes deep in the money, it becomes indistinguishable from a forward contract. If it is deep out of the money, it ceases to have value at all.

EXHIBIT 4 LOG CONTRACT IS EXCELLENT PROXY FOR VOLATILITY



positions rebalanced daily
recast volatility of 10.8%

An option purchaser must either accept the fact that the instrument may well become illiquid and cease to have time value, or else accept the costs and risks of rolling over the position periodically into another contract.

Log Contracts are quite different. There is only one contract for any given maturity. The contract keeps its sensitivity to volatility, whatever the asset price. It is therefore likely to retain its liquidity whatever happens to the price of the underlying. The Log Contract has merit as a derivative claim in its own right as well as an instrument designed specifically to hedge volatility.

APPENDIX

NOTATION

- t is the t 'th trading day. Both the futures and the Log Contract start at day 0 and expire at day T .
- F_t is the futures price on day t .
- L_t is the price of the Log Contract on day t ($L_T = \text{Log } F_T$).

B_t is the value of a deposit of 1 that is reinvested at the one-day riskless rate of interest from day 0 to day t ($B_0 = 1$).
 r_t is 1 + the interest rate from day t to day t + 1.
 W_t is the trader's cash position on day t ($W_0 = 0$).

Suppose that on day t, the trader goes short B_{t+1} Log Contracts and long B_{t+1}/F_t futures contracts. Wealth on day t + 1 is given by:

$$W_{t+1} = r_t W_t - B_{t+1} (L_{t+1} - L_t) + \frac{B_t}{F_t} (F_{t+1} - F_t)$$

If this is done from day 0 to day T, wealth on day T will be:

$$W_T = B_T \left\{ L_0 - \text{Log} F_0 + \sum_{t=1}^T [e^{x_t} - x_t - 1] \right\}$$

where $x_t \equiv \text{Log} (F_t/F_{t-1})$.

If jumps are not large, we can neglect terms of order x^3 , and write this as:

$$W_T \approx B_T \left\{ L_0 - \text{Log} F_0 + \frac{1}{2} \sigma^2 T \right\}$$

The terminal wealth is directly proportional to the squared outcome volatility.

ENDNOTES

¹Strictly speaking, this assumes that the forward price is an unbiased predictor of the future spot price, and that the implied volatility of the option price is an unbiased predictor of the outcome volatility.

²Outcome volatility is computed as the annualized root mean square log return rather than the annualized standard deviation of log returns. There are two reasons for computing the volatility this way. First, it correlates more closely with the hedge error than does the sample standard deviation. Second, if it is known that the drift is close to zero, the sample second moment gives a better estimate of the population variance than does the sample variance.

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